Int. J. Solids Structures, 1971, Vol. 7, pp. 1109 to 1123. Pergamon Press. Printed in Great Britain

THE STABILITY OF CYLINDRICAL SHELL UNDER GRAVITY LOAD

W. FLÜGGE

Stanford University, Stanford, California

and

GEORGE C. M. CHIANG

California State College, Fullerton, California

Abstract—In this paper, an analysis is made to study the stability of a cylindrical shell of finite length, placed horizontally and simply supported at both ends. The load on the shell is a simulation of its weight. The partial differential equations governing the stability of the shell contain variable coefficients x and ϕ. The solution to the differential equations is assumed in the form of a double Fourier series. The coefficient determinant of a set of infinitely many algebraic equations is of infinite order and the convergence of such determinant is proved for the case of doubly symmetrical buckling modes. Numerical results obtained with the use of a digital computer are presented to verify the convergence of the determinant and also to show the non-dimensional critical load.

INTRODUCTION

A BARREL-VAULT roof is a cylindrical shell with horizontal generators, carrying essentially its own weight. For several decades such structures have been built, but the problem of their elastic stability has defied solution. The present paper represents an effort in this direction. It deals with a complete circular cylinder, supported at its ends and loaded by its weight. To keep the complexity of the work within bounds while still getting at the essence of the problem, the actual load, uniform in the direction of the generators, has been replaced by the first term of its Fourier expansion. Beyond this simplification, the theory presented here is exact in the sense of linear stability theory.

DIFFERENTIAL EQUATIONS OF STABILITY

Before we derive the differential equations, let us introduce first the following definitions and considerations. The loads applied to the shell before buckling are called the basic loads, and the corresponding stress resultants and associated deformation are called the basic stress resultants and the basic deformation, respectively. A subscript I is used for the basic stress resultants to distinguish them from the additional forces appearing when the shell buckles. The basic deformation is considered to be small and is eliminated from our consideration by tracing the coordinate lines on the cylinder after this deformation has taken place. The displacements u, v, w are used to describe the additional deformation due to buckling.

The additional stress resultants due to buckling are denoted by $N_\phi, N_x, \ldots M_\phi, Q_x$. The normal and shearing forces are additional to the basic forces of the same kind. Therefore the total forces are

$$\bar{N}_\phi = N_{\phi I} + N_\phi, \qquad \bar{N}_x = N_{xI} + N_x,$$
$$\bar{N}_{x\phi} = N_{x\phi I} + N_{x\phi}, \qquad \bar{N}_{\phi x} = N_{\phi x I} + N_{\phi x}.$$

The stress resultants of a shell element in the buckled state are shown in Fig. 2. The introduction of the reference vectors $(1+\varepsilon_x)$ or $(1+\varepsilon_\phi)$ has been discussed by Flügge [1, p. 414]. They stem from the consideration of the law of conservation of energy in closed cycles of loading and unloading.

We shall now proceed to find the differential equations of stability. We assume a load $p\cos(\pi x/l)$ per unit of area, directed vertically downward. This is the first harmonic of an actual gravity load of intensity $p\pi/4$. This substitution has been chosen to limit the complexity of the mathematics to what appears to be most important for the result. With the boundary conditions that $N_{xI} = 0$ at $x = \pm l/2$, we obtain the following basic stress resultants:

$$N_{\phi I} = -pa\cos(\pi x/l)\cos\phi,$$
$$N_{x\phi I} = -(2pl/\pi)\sin(\pi x/l)\sin\phi,$$
$$N_{xI} = -(2p/a)(l/\pi)^2\cos(\pi x/l)\cos\phi.$$

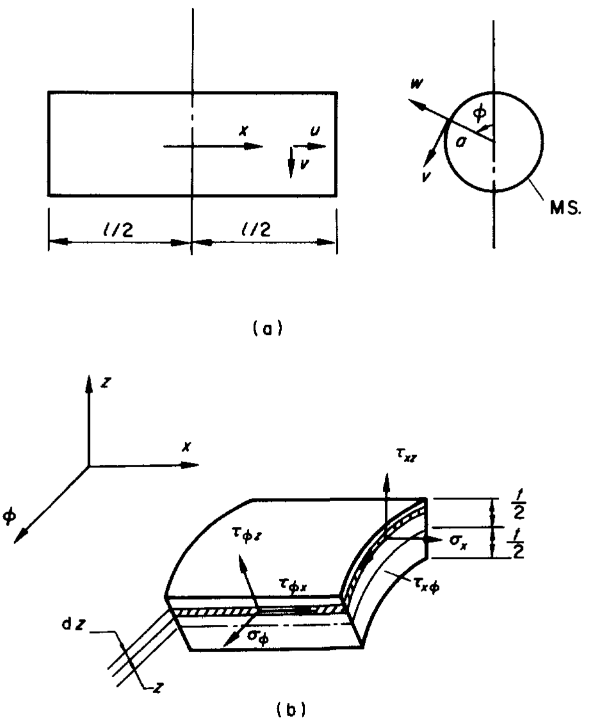


FIG. 1. Sign conventions for coordinates, displacements and stresses of a cylinder. (a) Coordinates and displacements. (b) Stresses.

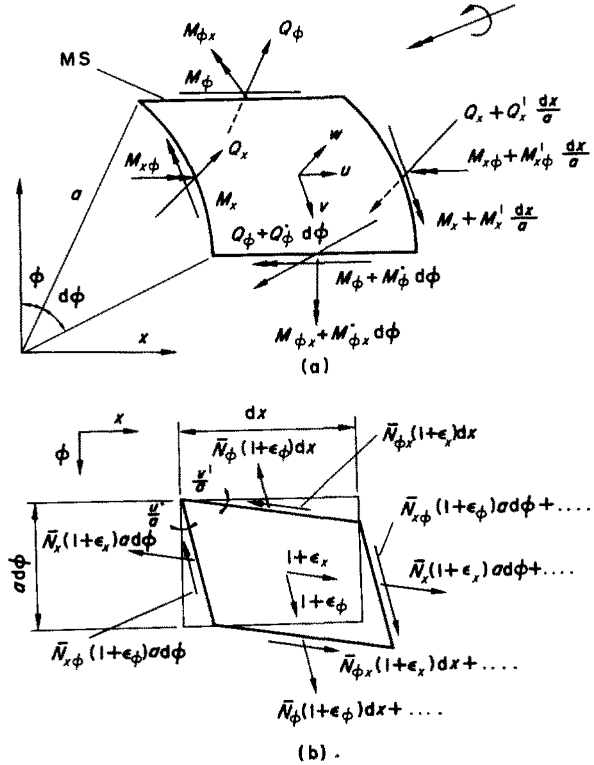


FIG. 2. Stress resultants of a shell element in the buckled state.

The six equations of equilibrium of a shell element as shown in Figs. 2(a) and (b) can now be obtained. The derivatives with respect to the dimensionless coordinates x/a and ϕ are indicated by primes and dots respectively, i.e. $a[\partial(\ )/\partial x] = (\ )'$ and $\partial(\ )/\partial\phi = (\ )^{\cdot}$. All the quadratic disturbance quantities such as $N'_x\varepsilon_x$ and $N_x\varepsilon'_x$ are neglected. Also, the cosine of a small angle is approximated by unity. The six equations are

$$N'_x + N^{\cdot}_{\phi x} - p\left\{\cos\frac{\pi x}{l}\cos\phi\left[2\left(\frac{l}{\pi a}\right)^2 u'' + u^{\cdot\cdot}\right] + \frac{4l}{\pi a}\sin\frac{\pi x}{l}\sin\phi\, u'^{\cdot} + \cos\frac{\pi x}{l}\sin\phi\, u^{\cdot}\right\} = 0,$$

$$N^{\cdot}_{\phi} + N'_{x\phi} - Q_{\phi} - p\left\{\cos\frac{\pi x}{l}\cos\phi\left[v^{\cdot\cdot} + 2\left(\frac{l}{\pi a}\right)^2 v'' + w^{\cdot}\right]\right.$$

$$\left. + 2\frac{l}{\pi a}\sin\frac{\pi x}{l}\sin\phi(2v'^{\cdot} + w') + \cos\frac{\pi x}{l}\sin\phi(v^{\cdot} + w)\right\} = 0,$$

$$Q^{\cdot}_{\phi} + Q'_x + N_{\phi} + p\left\{\cos\frac{\pi x}{l}\cos\phi\left[2\left(\frac{l}{\pi a}\right)^2 w'' + w^{\cdot\cdot} - 2v^{\cdot} - w\right]\right.$$

$$+ 2\frac{l}{\pi a}\sin\frac{\pi x}{l}\sin\phi(2w'^{\cdot} - v') + 2\cos\frac{\pi x}{l}\sin\phi(w^{\cdot} - v)$$

$$\left. + 2\frac{l}{\pi a}\sin\frac{\pi x}{l}\cos\phi\, w'\right\} = 0,$$

$$\dot{M}_{\phi}+M'_{x\phi}-aQ_{\phi} = 0, \qquad (1a–f)$$

$$M'_{x}+\dot{M}_{\phi x}-aQ_{x} = 0,$$

$$aN_{x\phi}-aN_{\phi x}+M_{\phi x} = 0.$$

By eliminating Q_x and Q_ϕ by means of equations (1d), (1e) and introducing the elastic law obtained by Flügge (see [1], p. 214) into the remaining equilibrium equations, we obtain the following three coupled linear homogeneous partial differential equations for u, v and w:

$$u''+\frac{1-\nu}{2}\ddot{u}+\frac{1+\nu}{2}\dot{v}'+\nu w'+k\left[\frac{1-\nu}{2}(\ddot{u}+\ddot{w}')-w'''\right]$$

$$-q\left[\cos\frac{\lambda x}{a}\cos\phi\left(\frac{2}{\lambda^2}u''+\ddot{u}\right)+\frac{4}{\lambda}\sin\frac{\lambda x}{a}\sin\phi\, \dot{u}'+\cos\frac{\lambda x}{a}\sin\phi\, \dot{u}\right] = 0,$$

$$\frac{1+\nu}{2}\dot{u}'+\ddot{v}+\frac{1-\nu}{2}v''+\dot{w}+k\left[\frac{3}{2}(1-\nu)v''-\frac{3-\nu}{2}\dot{w}''\right]$$

$$-q\left[\cos\frac{\lambda x}{a}\cos\phi\left(\ddot{v}+\frac{2}{\lambda^2}v''+\dot{w}\right)+\frac{2}{\lambda}\sin\frac{\lambda x}{a}\sin\phi(2\dot{v}'+w')\right.$$

$$\left.+\cos\frac{\lambda x}{a}\sin\phi(\dot{v}+w)\right] = 0, \qquad (2a–c)$$

$$\nu u'+\dot{v}+w+k\left[\frac{1-\nu}{2}\ddot{u}'-u'''-\frac{3-\nu}{2}\dot{v}''+w''''+2\ddot{w}''+w^{\cdot\cdot\cdot\cdot}+2\ddot{w}+w\right]$$

$$+q\left[\cos\frac{\lambda x}{a}\cos\phi\left(\frac{2}{\lambda^2}w''+\ddot{w}-2\dot{v}-w\right)\right.$$

$$\left.+\frac{2}{\lambda}\sin\frac{\lambda x}{a}\sin\phi(2\dot{w}'-v')+2\cos\frac{\lambda x}{a}\sin\phi(\dot{w}-v)+\frac{2}{\lambda}\sin\frac{\lambda x}{a}\cos\phi w'\right] = 0.$$

The dimensionless parameters in these equations are

$$k = \frac{K}{Da^2} = \frac{t^2}{12a^2}, \qquad q = \frac{pa}{D}, \qquad \lambda = \frac{\pi a}{l},$$

and the quantities $D = Et/(1-\nu^2)$ and $K = Et^3/12(1-\nu^2)$ are, respectively, the extensional and flexural rigidities of the shell.

SOLUTION OF THE DIFFERENTIAL EQUATIONS

The assumed solution which will represent the doubly symmetric buckling mode, can be written in the following infinite series:

$$
\begin{aligned}
u &= \sum_{m=0,1,2,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} A_{mn} \cos m\phi \sin \frac{n\lambda x}{a}, \\
v &= \sum_{m=1,2,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} B_{mn} \sin m\phi \cos \frac{n\lambda x}{a}, \\
w &= \sum_{m=0,1,2,\ldots}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} C_{mn} \cos m\phi \cos \frac{n\lambda x}{a}.
\end{aligned}
\tag{3}
$$

Because of the restriction of n to odd values, the solution (3) satisfies the usual (and useful) boundary conditions

$$v = w = 0, \qquad N_x = M_x = 0$$

at both ends of the cylinder.

When the solution (3) is introduced into the differential equations (2a–c), there appear products of two trigonometric functions of the same coordinate. Well known identities like

$$\cos(n\lambda x/a)\cos(\lambda x/a) = \frac{1}{2}\left[\cos\frac{(n-1)\lambda x}{a} + \cos\frac{(n+1)\lambda x}{a}\right]$$

may be applied, but these lead to the appearance of even values of n, which are undesirable. They can be avoided by expanding each sine or cosine of such an argument into a series of sines or cosines of $(n+\omega)\lambda x/a$ with odd n and odd $(n+\omega)$. After this has been done, the right-hand side of each equation is a double Fourier series, using all integers m and all odd integers n. To make these series vanish identically the following equations must hold:

$$
\begin{aligned}
&A_{mn}a^{11}_{mn,mn} + B_{mn}a^{12}_{mn,mn} + C_{mn}a^{13}_{mn,mn} \\
&\quad + \frac{q}{2\pi} \sum_{\omega=0,\pm 2,\ldots} [A_{m-1,n+\omega}a^{11}_{mn,m-1,n+\omega} + A_{m+1,n+\omega}a^{11}_{mn,m+1,n+\omega}] = 0, \\
&A_{mn}a^{21}_{mn,mn} + B_{mn}a^{22}_{mn,mn} + C_{mn}a^{23}_{mn,mn} \\
&\quad + \frac{q}{2\pi} \sum_{\omega=0,\pm 2,\ldots} [B_{m-1,n+\omega}a^{22}_{mn,m-1,n+\omega} + B_{m+1,n+\omega}a^{22}_{mn,m+1,n+\omega} \\
&\quad + C_{m-1,n+\omega}a^{23}_{mn,m-1,n+\omega} + C_{m+1,n+\omega}a^{23}_{mn,m+1,n+\omega}] = 0, \\
&A_{mn}a^{31}_{mn,mn} + B_{mn}a^{32}_{mn,mn} + C_{mn}a^{33}_{mn,mn} \\
&\quad + \frac{q}{2\pi} \sum_{\omega=0,\pm 2,\ldots} [B_{m-1,n+\omega}a^{32}_{mn,m-1,n+\omega} + B_{m+1,n+\omega}a^{32}_{mn,m+1,n+\omega} \\
&\quad + C_{m-1,n+\omega}a^{33}_{mn,m-1,n+\omega} + C_{m+1,n+\omega}a^{33}_{mn,m+1,n+\omega}] = 0, \\
&\quad m = 0, 1, 2, \ldots, n = 1, 3, 5, \ldots.
\end{aligned}
\tag{4a–c}
$$

The first superscript of the coefficients in the above equations denotes the order of the equation in the triplet. In the second superscript, the numbers 1, 2, 3 distinguish coefficients of A, B, C, respectively. The first two subscripts denote the particular indices m and n given to the triplet and the last two subscripts agree with the subscripts of the unknown quantity A, B, or C. These coefficients are expressed as follows:

$$a^{11}_{mnmn} = (n\lambda)^2 + \frac{1-\nu}{2}(1+k)m^2,$$

$$a^{12}_{mnmn} = a^{21}_{mnmn} = \frac{1+\nu}{2}\lambda(mn),$$

$$a^{13}_{mnmn} = a^{31}_{mnmn} = \nu(n\lambda) + k\left[(n\lambda)^3 - \frac{1-\nu}{2}(n\lambda)m^2\right],$$

$$a^{22}_{mnmn} = m^2 + \frac{1-\nu}{2}(1+3k)(n\lambda)^2,$$

$$a^{23}_{mnmn} = a^{32}_{mnmn} = m\left[1 + \frac{k(3-\nu)}{2}(n\lambda)^2\right],$$

$$a^{33}_{mnmn} = 1 + k[(n\lambda)^4 + 2(n\lambda)^2 m^2 + m^4 - 2m^2 + 1],$$

and for $n+\omega = 1$:

$$a^{11}_{mn,m-1,1} = \left[-\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][2+(m-1)^2 - 5(m-1)],$$

$$a^{11}_{mn,m+1,1} = \left[-\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][2+(m+1)^2 + 5(m+1)],$$

$$a^{22}_{mn,m-1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][2+(m-1)^2 - 5(m-1)],$$

$$a^{22}_{mn,m+1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][2+(m+1)^2 + 5(m+1)],$$

$$a^{23}_{mn,m-1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right](m-4),$$

$$a^{23}_{mn,m+1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right](m+4),$$

$$a^{32}_{mn,m-1,1} = \left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right](6-2m),$$

$$a^{32}_{mn,m+1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right](6+2m),$$

$$a^{33}_{mn,m-1,1} = \left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][-(m-1)^2 + 6(m-1) - 1],$$

$$a^{33}_{mn,m+1,1} = -\left[\frac{\sin(2-n)(\pi/2)}{2-n} + \frac{\sin(2+n)(\pi/2)}{2+n}\right][(m+1)^2 + 6(m+1) + 1],$$

and for $n+\omega \neq 1$, with

$$H_1 = \frac{\sin(\omega-1)(\pi/2)}{\omega-1}, \qquad H_2 = \frac{\sin(2n+\omega-1)(\pi/2)}{2n+\omega-1},$$

$$H_3 = \frac{\sin(\omega+1)(\pi/2)}{\omega+1}, \qquad H_4 = \frac{\sin(2n+\omega+1)(\pi/2)}{2n+\omega+1}:$$

$$a^{11}_{mn,m-1,n+\omega} = (-H_1+H_2)[2(n+\omega)^2+(m-1)^2-(m-1)+4(n+\omega)(m-1)]$$
$$+(-H_3+H_4)[2(n+\omega)^2+(m-1)^2-(m-1)-4(n+\omega)(m-1)],$$

$$a^{11}_{mn,m+1,n+\omega} = (-H_1+H_2)[2(n+\omega)^2+(m+1)^2+(m+1)-4(n+\omega)(m+1)]$$
$$+(-H_3+H_4)[2(n+\omega)^2+(m+1)^2+(m+1)+4(n+\omega)(m+1)],$$

$$a^{22}_{mn,m-1,n+\omega} = -(H_1+H_2)[2(n+\omega)^2+(m-1)^2-(m-1)+4(n+\omega)(m-1)]$$
$$-(H_3+H_4)[2(n+\omega)^2+(m-1)^2-(m-1)-4(n+\omega)(m-1)],$$

$$a^{22}_{mn,m+1,n+\omega} = -(H_1+H_2)[2(n+\omega)^2+(m+1)^2+(m+1)-4(n+\omega)(m+1)]$$
$$-(H_3+H_4)[2(n+\omega)^2+(m+1)^2+(m+1)+4(n+\omega)(m+1)],$$

$$a^{23}_{mn,m-1,n+\omega} = -(H_1+H_2)[2(n+\omega)+(m-2)]-(H_3+H_4)[-2(n+\omega)+(m-2)],$$

$$a^{23}_{mn,m+1,n+\omega} = -(H_1+H_2)[-2(n+\omega)+(m+2)]-(H_3+H_4)[2(n+\omega)+(m+2)],$$

$$a^{32}_{mn,m-1,n+\omega} = (H_1+H_2)[-2(n+\omega)-2(m-1)+2]+(H_3+H_4)[2(n+\omega)-2(m-1)+2],$$

$$a^{32}_{mn,m+1,n+\omega} = (H_1+H_2)[2(n+\omega)-2(m+1)-2]+(H_3+H_4)[-2(n+\omega)-2(m+1)-2],$$

$$a^{33}_{mn,m-1,n+\omega} = (H_1+H_2)[-2(n+\omega)^2-(m-1)^2-1-4(n+\omega)(m-1)-2(n+\omega)+2(m-1)]$$
$$+(H_3+H_4)[-2(n+\omega)^2-(m-1)^2-1+4(n+\omega)(m-1)+2(n+\omega)-2(m-1)],$$

$$a^{33}_{mn,m+1,n+\omega} = (H_1+H_2)[-2(n+\omega)^2-(m+1)^2-1+4(n+\omega)(m+1)-2(n+\omega)-2(m+1)]$$
$$+(H_3+H_4)[-2(n+\omega)^2-(m+1)^2-1-4(n+\omega)(m+1)$$
$$+2(n+\omega)-2(m+1)].$$

CONVERGENCE PROOF FOR THE COEFFICIENT DETERMINANT

Equations (4a–c) represent a set of infinitely many homogeneous equations in the infinitely many unknowns A_{mn}, B_{mn} and C_{mn}. A finite system of homogeneous equations has a nontrivial solution if and only if its coefficient determinant vanishes. The same can be true for infinite systems if the determinant of infinite order is meaningful, i.e. if there exists a convergent process defining the value of the determinant. The convergence of the coefficient determinant of equations (4a–c) is proved for the reduced system which is obtained by eliminating all unknowns A_{mn} and B_{mn} from equations (4a–c). During the process of elimination, terms with a factor q^2 are neglected as being small of higher order.

The final equation for the unknown C is as follows:

$$\begin{vmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{vmatrix}_{mnmn} . C_{mn}+\frac{q}{2\pi}[(a^{31}a^{12}-a^{11}a^{32})_{mnmn}M$$

$$+(a^{21}a^{32}-a^{22}a^{31})_{mnmn}N+(a^{11}a^{22}-a^{12}a^{21})_{mnmn}Q] = 0. \tag{5}$$

In equation (5), the subscripts at the end of a parenthesis or determinant apply to all quantities inside and the terms M, N and Q stand for the following expressions:

$$M = \sum_{\omega}\Bigg[C_{m-1,n+\omega}a^{22}_{mn,m-1,n+\omega}\left(\frac{a^{13}a^{21}-a^{11}a^{23}}{\Delta}\right)_{m-1,n+\omega,m-1,n+\omega}$$

$$+C_{m+1,n+\omega}a^{22}_{mn,m+1,n+\omega}\left(\frac{a^{13}a^{21}-a^{11}a^{23}}{\Delta}\right)_{m+1,n+\omega,m+1,n+\omega}$$

$$+C_{m-1,n+\omega}a^{23}_{mn,m-1,n+\omega}+C_{m+1,n+\omega}a^{23}_{mn,m+1,n+\omega}\Bigg],$$

$$N = \sum_{\omega}\Bigg[C_{m-1,n+\omega}a^{11}_{mn,m-1,n+\omega}\left(\frac{a^{12}a^{23}-a^{13}a^{22}}{\Delta}\right)_{m-1,n+\omega,m-1,n+\omega}$$

$$+C_{m+1,n+\omega}a^{11}_{mn,m+1,n+\omega}\left(\frac{a^{12}a^{23}-a^{13}a^{22}}{\Delta}\right)_{m+1,n+\omega,m+1,n+\omega}\Bigg],$$

$$Q = \sum_{\omega}\Bigg\{C_{m-1,n+\omega}\left[a^{33}_{mn,m-1,n+\omega}+a^{32}_{mn,m-1,n+\omega}\left(\frac{a^{13}a^{21}-a^{11}a^{23}}{\Delta}\right)_{m-1,n+\omega,m-1,n+\omega}\right]$$

$$+C_{m+1,n+\omega}\left[a^{33}_{mn,m+1,n+\omega}+a^{32}_{mn,m+1,n+\omega}\left(\frac{a^{13}a^{21}-a^{11}a^{23}}{\Delta}\right)_{m+1,n+\omega,m+1,n+\omega}\right]\Bigg\},$$

where

$$\Delta = \begin{vmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{vmatrix} = a^{11}a^{22}-a^{12}a^{21}.$$

With the substitution of M, N and Q in equation (5) this equation assumes the following form:

$$C_{mn}\cdot\begin{vmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{vmatrix}_{mnmn} +\frac{q}{2\pi}\sum_{\omega}[\ldots C_{m+1,n+\omega}+\ldots C_{m-1,n+\omega}] = 0. \tag{6}$$

The coefficients of $C_{m+1,n+\omega}$ and $C_{m-1,n+\omega}$, which have been indicated by a row of dots, are complicated expressions depending on m, n and ω.

Our objective here is to find a way to write down equation (6) for all values of m and n such that the determinant of the coefficients may be written as

$$\text{Det} = \begin{vmatrix} 1+b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & 1+b_{22} & b_{23} & \cdots \\ b_{31} & b_{32} & 1+b_{33} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{7}$$

A sufficient condition for the convergence of the above infinite determinant is to have $\Sigma_{i,j} b_{ij}^2$ absolutely convergent [2, pp. 42–43].

In the convergence proof, we are interested mainly in the order of magnitude of the coefficients when we let both m and n go to infinity. Therefore, we shall only concern ourselves with the governing terms, i.e. those which are of the highest degree in m and n. For this reason, we rewrite equation (6) with the coefficients of the unknowns expressed only in their order of magnitude. They are as follows:

$$[(m^2+n^2)^4+(m^2+n^2)^3+(m^2+n^2)^2]C_{mn}$$
$$+\frac{q}{2\pi}\left\{\ldots n(m^2+n^2)\sum_{\omega}[(n+\omega)^2+(m\pm 1)^2\ldots]\frac{(n+\omega)}{[(n+\omega)^2+(m\pm 1)^2]}C_{m\pm 1,n+\omega}\ldots\right\} = 0. \tag{8}$$

In obtaining this equation, we have neglected terms with $q \, . \, k$ as being small. We have also neglected terms such as km^4 when compared with m^2. This can be done for the following reason. In practical problems, the geometrical parameter k is a very small quantity. Thus in order to have km^4 comparable to m^2, the index m has to be very large such that the wavelength in the circumferential direction is only a small multiple of the shell thickness. When this happens, the linear theory of the *thin* shell is no longer valid. Furthermore, to produce such small waves requires a high t/a ratio and a very short span. The corresponding critical load in this case will be so large that it will never occur in the real application. Therefore, within our present theory and also considering the practicality, such simplification is entirely justified.

The $\pm$ sign inside the bracket of equation (8) corresponds to the $\pm$ sign appearing in the subscript of C. The coefficient of C_{mn} is an eighth-degree polynomial in both m and n. The expression of $(m^2+n^2)^4$ inside the coefficient bracket can be written as:

$$(m^4n^2+2m^2n^4+n^6)\left(\frac{m^4}{n^2}+2m^2+n^2\right).$$

Now, if we divide the entire equation (8) by

$$(m^4n^2+2m^2n^4+n^6) \text{ and use } \left(\frac{m^4}{n^2}+2m^2+n^2\right)C_{mn} = C'_{mn}$$

as a new unknown, we arrive at the following equation:

$$\left[1+\frac{(m^2+n^2)^3+(m^2+n^2)^2}{(m^2+n^2)^4}\right]C'_{mn}$$

$$+\frac{q}{2\pi}\left\{\ldots\frac{n(m^2+n^2)}{(m^4n^2+2m^2n^4+n^6)}\right.$$

$$\left.\times\sum_{\omega}\frac{(n+\omega)[(n+\omega)^2+(m\pm 1)^2+\ldots]}{\left[(n+\omega)^2+\frac{(m\pm 1)^4}{(n+\omega)^2}+2(m\pm 1)^2\right][(n+\omega)^2+(m\pm 1)^2]}C'_{m\pm 1,n+\omega}+\ldots\right\}=0. \quad (9)$$

For every admissible pair m, n [see equations (4)], we can write an equation of this sort. In every equation there are infinitely many terms due to the summation over ω. When writing these equations, we must choose a certain order in which they shall be listed. The order to be employed here is shown in Fig. 3. The double lines indicate the diagonal coefficients which are the coefficients of C_{mn} in equation (8) or those of C'_{mn} in equation (9). The single lines indicate the other non-vanishing coefficients as the result of the summation over ω.

We can now compare Fig. 3 with the determinant (7). The only difference between the two is that the elements in (7) are specified by only two indices, i and j, say, while in Fig. 3 they are specified by m, n in the column and $n+\omega$, $m\pm 1$ in the row. Nevertheless, the specification of an absolutely convergent summation of all the coefficients, individually squared, is still the condition for the convergence of the infinite determinant of Fig. 3 regardless how its elements are generated. Therefore, all we have to show is that the sum of the squares of the diagonal terms in Fig. 3—not including the numeral 1 which appears in equation (9)—and of the squares of the off diagonal terms is absolutely convergent.

Consider first all the terms which are off the diagonal, i.e. all the terms containing the load parameter q. From equation (9), we obtain the following sum of the squares of the coefficients after some factorization in the numerator and in the denominator:

$$\sum_n\sum_m\frac{1}{n^2m^4}\times\frac{1+\frac{2n^2}{m^2}+\frac{n^4}{m^4}}{1+\frac{2n^4}{m^4}+\frac{n^8}{m^8}+\ldots}\sum_{\omega}\frac{1+\frac{2n}{\omega}+\frac{n^2}{\omega^2}}{\omega^2\left(1+\frac{n^4}{\omega^4}+\frac{m^4}{\omega^4}+\frac{m^2n^2}{\omega^4}+\ldots\right)}.$$

The dots in the denominator indicate some complicated expressions of m and n whose degrees are lower than that of the terms shown, all divided by ω to a certain power. Therefore, they are of no significance.

If we now let ω, m, n go to infinity in this order, the summation converges as

$$\sum_n^{\infty}\frac{1}{n^2}\sum_m^{\infty}\frac{1}{m^4}\sum_{\omega}^{\infty}\frac{1}{\omega^2},$$

for large m, n and ω.

Next we shall consider all the terms which are on the diagonal. They are of the type $[(m^2+n^2)^3+(m^2+n^2)^2]/(m^2+n^2)^4$ which, for $m, n\to\infty$, is of the order $1/(m^2+n^2)$. When

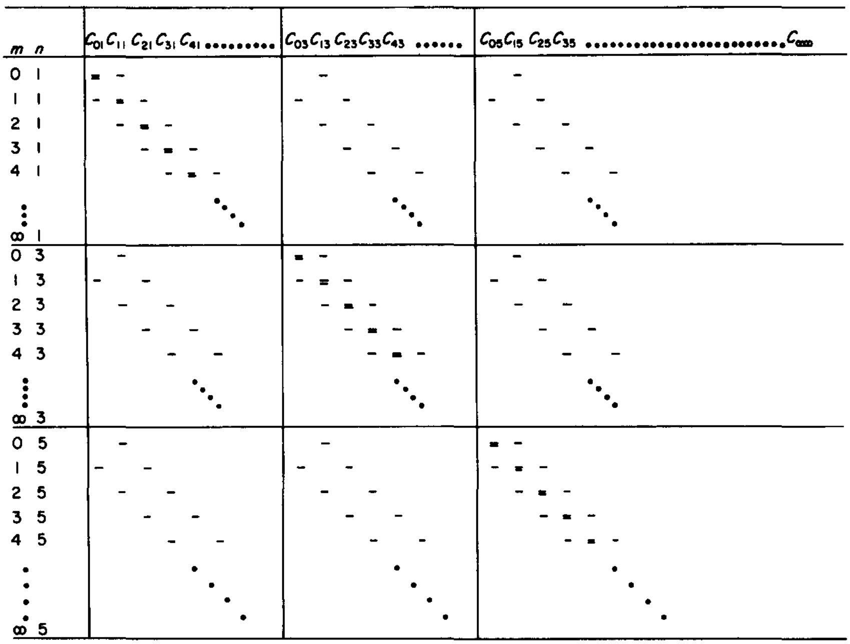


FIG. 3. Coefficient of C_{mn}.

we sum the square of it over m and n, we obtain the expression of

$$\sum_n \sum_m \frac{1}{(m^2+n^2)^2}.$$

This series is term by term less than the series

$$\sum_n \sum_m \frac{1}{m^4+n^4},$$

which is found [3, pp. 22–23] to be equal to

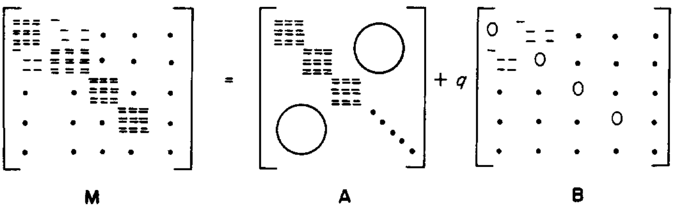
$$\sum_n^{\infty} \sum_m^{\infty} \frac{1}{(m^2)^2+(n^2)^2} = \sum_n \left[\frac{\pi}{2n^2} \coth \pi n^2 - \frac{1}{2n^4}\right],$$

and this converges like $\Sigma_n 1/n^2$. Therefore our series $\Sigma_n \Sigma_m 1/(m^2+n^2)^2$ is convergent by comparison.

The convergence of the infinite-order coefficient determinant is now proved. The success of this process relies on the thorough study of all the parameters contained in the coefficients, the proper grouping and factorization of all the indices involved, and the proper arrangement of all the summations so that the dependence of one index upon another is not destroyed.

NUMERICAL RESULTS

Since we have proved the convergence of the infinite determinant, we can now take out a finite segment for the approximate numerical evaluation of the critical load q. The equations (4a–c) will be used directly. The matrix is shown schematically in Fig. 4. The double lines indicate the coefficients of A_{mn}, B_{mn} and C_{mn}, which do not contain the load term q. The single lines indicate the coefficients which contain the load term q. Let us denote by **M** the matrix shown in Fig. 4. It is clear that this matrix can be decomposed into two matrices **A** and **B** as shown in the following:



If **x** is a column vector representing all the unknowns A_{mn}, B_{mn} and C_{mn} shown in Fig. 4, we can write equations (4a–c) in the following matrix form:

$$(\mathbf{A}+q\mathbf{B})\mathbf{x} = \mathbf{0}. \tag{10}$$

With division by q and premultiplication by $\mathbf{A}^{-1}$, equation (10) takes the form

$$(\mathbf{C}-\Omega\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{11}$$

where

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}, \qquad \text{and} \qquad \Omega = -\frac{1}{q}.$$

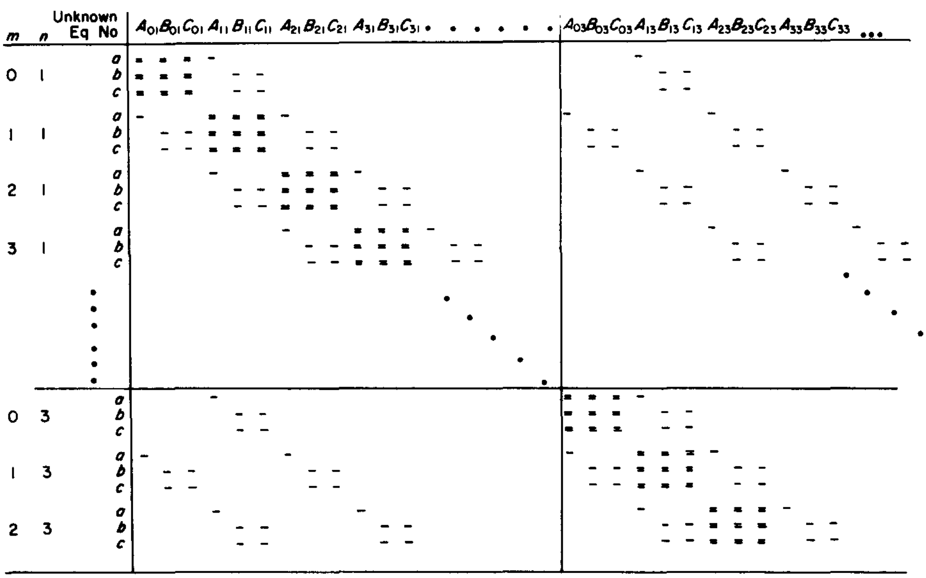


FIG. 4.

Because we have chosen a particular way to arrange the unknowns as shown in Fig. 3, the matrix **C** is neither symmetric nor positive definite. The eigenvalues Ω, therefore, will not all be real and positive. For the physical problem one is interested only in the highest, real, negative Ω which corresponds to the lowest, real and positive q.

The numerical work done consisted of two parts. The first part was to verify the convergence of the eigenvalue numerically for one set of parameters. The second part was to obtain a set of approximate eigenvalues for varying parameters of the shell. To check upon the convergence, the following values of the shell parameters were chosen:

$$t/a = 0{\cdot}01, \qquad \lambda = \pi a/l = 0{\cdot}25 \quad \text{and} \quad \nu = 0{\cdot}25.$$

A computer program was written for finding all the eigenvalues. The calculations were made on the Burroughs 5500 electronic digital computer of Stanford University. The highest negative Ω's from determinants of different sizes are tabulated in Tables 1 and 2. Table 1 shows the results for a fixed choice of values m (referring to the waves in circumferential direction) and with an increasing number of n values. The tendency to converge rapidly for an increasing number of n's can be seen from the ratio of the successive Ω's shown in the last column. Table 2 shows the results for a fixed choice of n's with an increasing number of m's. The convergence is also quite good in this case. The results of Tables 1 and 2 have been plotted in Fig. 5. The dominating components of the buckling mode for the shell at hand are seen to be $n = 3$ and $m = 3, 4, 5, 6$ as one might expect for a very long shell.

The set of curves plotted in Fig. 6 has been obtained for a fixed choice $n = 3, 5, 7$ and $m = 3, 4, 5, 6$. The eigenvalue will become less accurate as the parameter λ increases. Accurate result for large λ can be obtained if a larger computer is available for handling a determinant of high order.

TABLE 1. VARIATION OF HIGHEST EIGENVALUE WITH NUMBER OF LONGITUDINAL WAVES

$\lambda = 0.25$, $m = 3, 4, 5$

i	n	Ω_i	Ratio Ω_{i+1}/Ω_i
1	1	$-1{\cdot}714$	
			2840·00
2	1, 3	$-4{\cdot}8804 \times 10^3$	
			1·0618
3	1, 3, 5	$-5{\cdot}1824165 \times 10^3$	
			1·0158
4	1, 3, 5, 7	$-5{\cdot}26475 \times 10^3$	
			1·0046
5	1, 3, 5, 7, 9	$-5{\cdot}28923 \times 10^3$	

TABLE 2. VARIATION OF HIGHEST EIGENVALUE WITH NUMBER OF CIRCUMFERENTIAL WAVES

$\lambda = 0{\cdot}25$, $n = 1, 3$

i	m	Ω_i	Ratio Ω_{i+1}/Ω_i
1	3, 4, 5	$-4{\cdot}8804 \times 10^3$	
			1·0552
2	3, 4, 5, 6	$-5{\cdot}14996 \times 10^3$	
			1·0032
3	2, 3, 4, 5, 6, 7, 8	$-5{\cdot}16646 \times 10^3$	

CONCLUDING REMARKS

The above theoretical investigation of the elastic stability of a cylinder carrying a gravity load is based on the linear theory. Although it is known that the critical load from the linear theory is usually higher than those obtained from the experiments due to many other factors, nevertheless it can still serve as an upper bound. The stability of the cylindrical segments widely used as shell roofs is still an unsolved problem. The difficulty is mainly due to the complicated prebuckling stresses caused by the edge disturbances. The results of this work may serve as an estimate of the stability of such a shell.

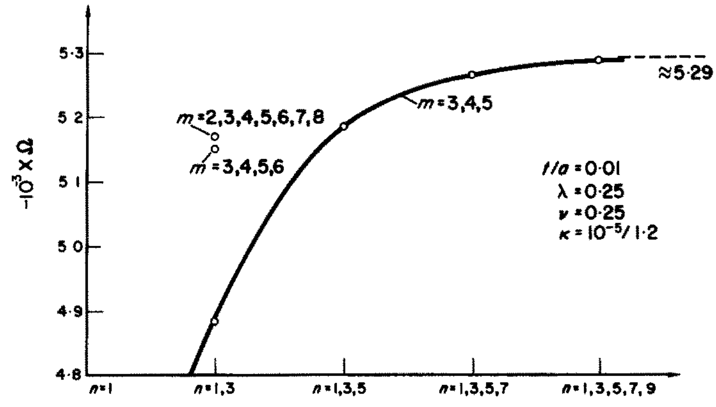


FIG. 5.

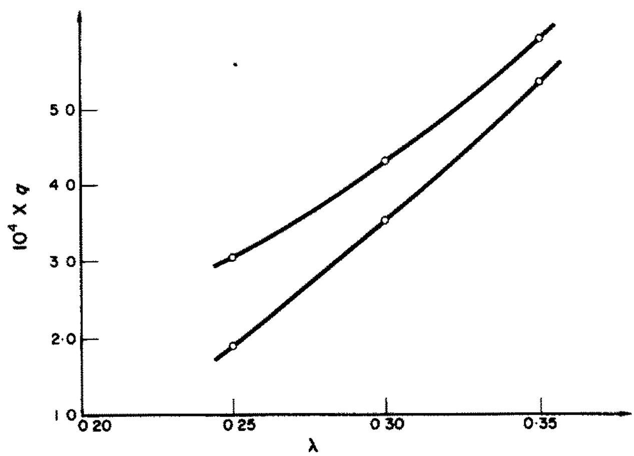


FIG. 6.

REFERENCES

[1] W. FLÜGGE, *Stresses in Shells*. Springer-Verlag (1960).
[2] L. V. KANTOROVICH and V. I. KRYLOV, *Approximate Methods of Higher Analysis*, translated by C. D. BENSTER. P. Noordhoff (1958).
[3] L. B. W. JOLLY, *Summation of Series*. Dover (1961).

(*Received* 13 *May* 1970)

Абстракт—В работе дается анализ для исследования устойчивости цилиндрической оболочки конечной длины, расположенной горизонтально и свободно опертой на двух краях. Нагрузкой оболочки является имитация ее веса. Частные дифференциальные уравнения, касающиеся устойчивости оболочки, заключают переменные коэффициенты x и ϕ. Решение дифференциальных уравнений применяется в форме двойных рядов фурье. Детерминант коэффициентов системы бесконечно многих алпебраических уравнеий является бесконечным. Доказывается сходимость макою детерминанта для случая двойных симметрических форм выпучивания. Дается численные результаты, полученные с помощью ЦВМ, чтобы проверить сходимость детерминанта, а также для того, чтобы указать безразмерную критическую нагрузку.